SCALING in
MECHANICAL MICRO-SYSTEMS

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Micro-606

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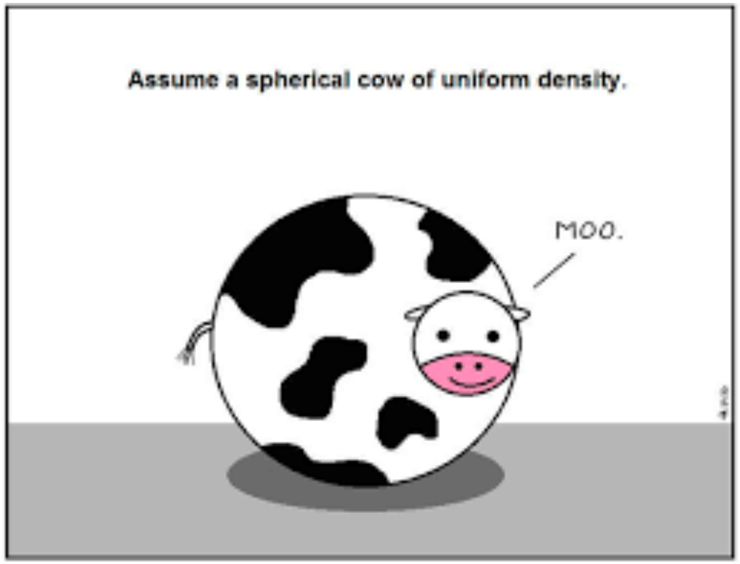
Mechanical Scaling in MEMS

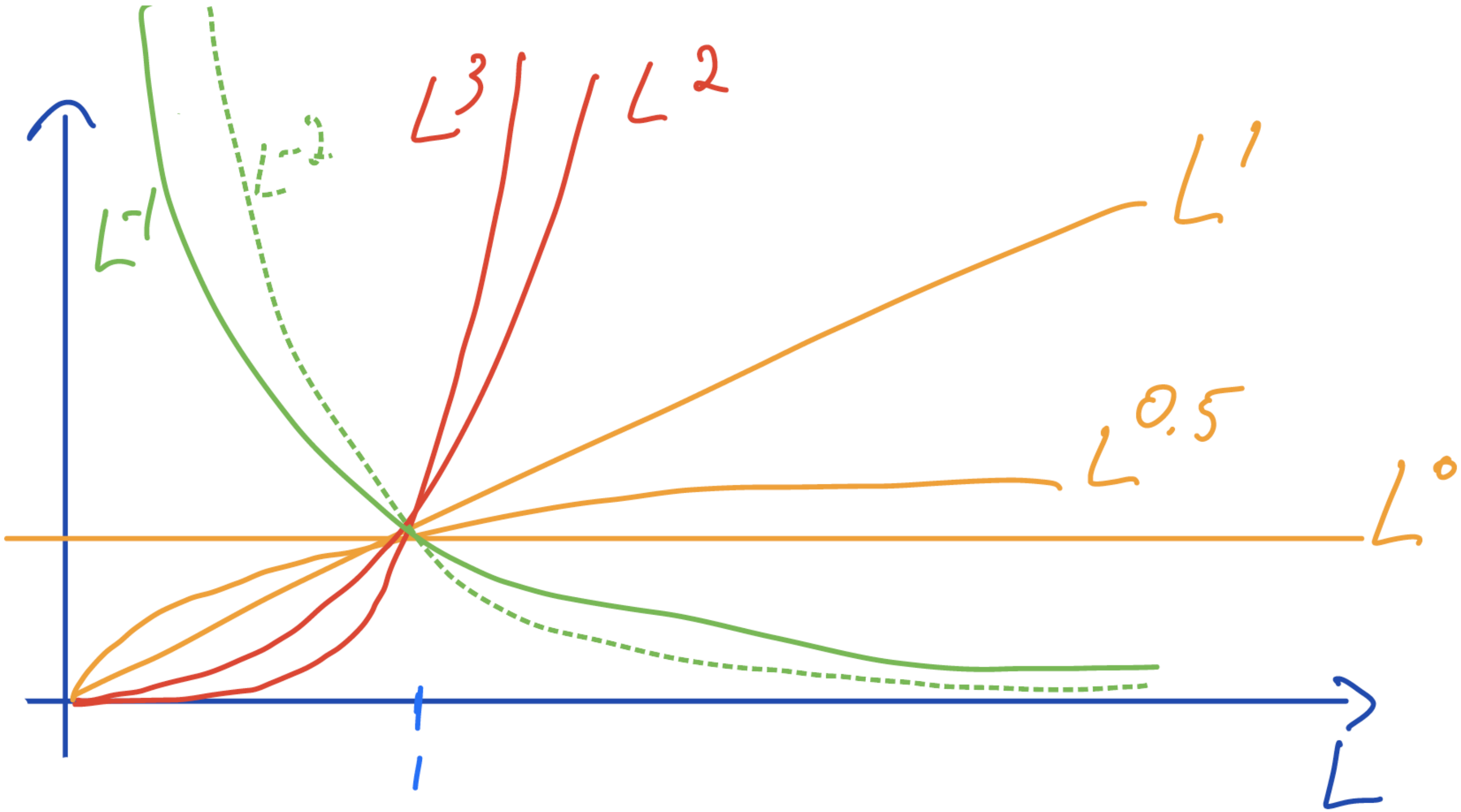
1. Intro: Scaling in Animals
2. Lumped Element Modeling
3. Cantilevers: modes, stress, stress gradient
4. Non-linearities in MEMS and the Duffing equation
5. Thermo-mechanical Noise

Simple scaling: parameter *L*

- One single parameter to describe all dimensions: ***L***
- Goal is overall rough scaling laws: what physical principles dominate at different length scale
- eg mass: $M \propto L^3$

  surface: $A \propto L^2$





L^3
L^2
L^1
$L^{0.5}$
L^0
L^{-1}
L^{-2}
1
L

Simplified overview of <u>mechanical</u> scaling laws. 1 of 3

Parameter	Scaling Law	comment
Mass	$M \propto L^3$	
Area	$A \propto L^2$	
Surface-to-volume ratio	$\gamma \propto L^{-1}$	=> good for chemical reactions
Inertial forces	$F_{inertial} \propto M \propto L^3$	
Contact forces	$F_{contact} \propto A \propto L^2$	
Contact/inertial forces ratio	$\propto L^{-1}$	=> bad for manipulation
Van der Waals forces	$F_{VdW} \propto d^{-7}$	very short range!

Simplified overview of mechanical scaling laws: 2 of 3

Parameter	Scaling Law	comment
Spring Constant	$k \propto L$	calculated using Hook's law on a bar of cross-section A and length l: $k = \frac{AE}{l}$ (E: Young's modulus)
=> Spring (restoring) forces	$F \propto kx \propto L^2$	
Acceleration (intrinsic)	$a = \frac{F}{M} \propto L^{-1}$	
Natural frequency	$\omega_0 = \sqrt{\frac{k}{M}} \propto L^{-1}$	
"switching" time	$t_s = 1/\omega_0 \propto L$	
Viscous drag forces	$F_{vd} \propto \eta L v \propto L$	
Quality factor (=energy stored/energy loss per cycle)	$Q_f \propto \frac{kx^2}{F_f x_0} \propto L$	

Simplified overview of mechanical scaling laws: 3 of 3: energy

Parameter	Scaling Law	comment
Kinetic Energy	$E_{kin} = \frac{mv^2}{2} \propto L^3$	assuming v is constant. If $v \propto L$, then $E_{kin} \propto L^5$
Potential energy of a spring	$E_{pot,spring} = \frac{1}{2}kx^2 \propto L^3$	
Mechanical power	$P_{mec} = Fv \propto L^2$	
Mechanical energy density	$\frac{E_{pot,spring}}{m} = \frac{1}{2}\frac{\sigma^2}{E} \propto L^0$	scale invariant

1. Intro: Scaling in Animals

EPFL

In movies: isometric scaling

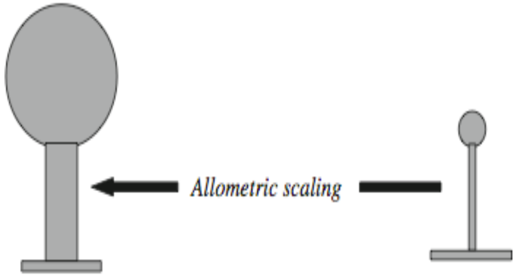
antman

1. In nature, allometric Scaling

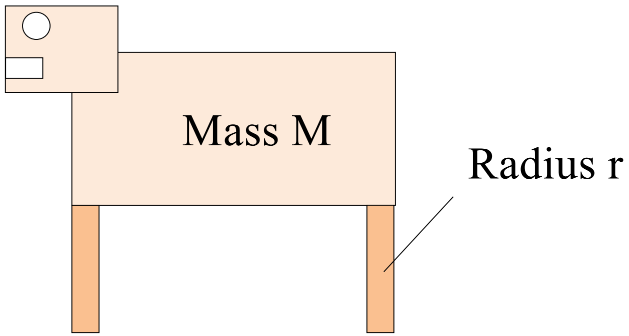
40 cm diameter

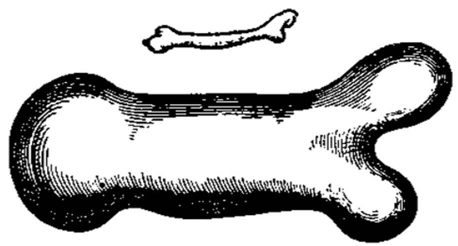
40 µm diameter

Aspect ratio changes when scale animals



Scaling of animal bone diameter





Galileo's depiction of the bones of light and heavy animals. (From *Dialogue on Two New Sciences*.) 1638

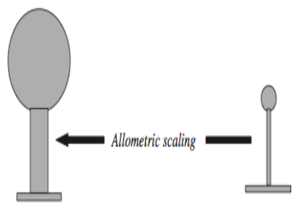
Assumptions:

- All animals have same bone material that fracture at a critical stress σ_{cr}
- Leg bone radius *r* is as small as possible while not fracturing from own weight

$$\sigma_{cr}\pi r^2 \backsim mg$$

$$r \propto \sqrt{m} \propto L^{3/2}$$

- This type of allometric scaling is a widely seen when miniaturizing mechanical systems

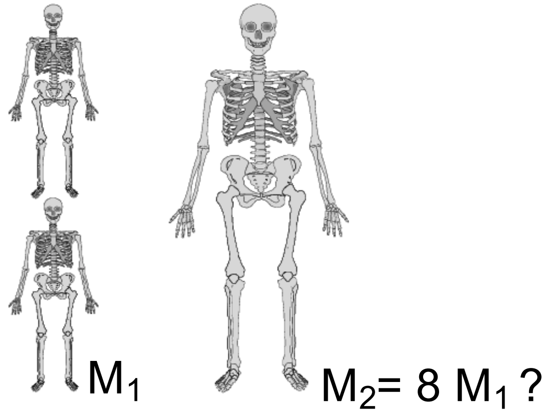


As it gets bigger, the animal eventually is only bone…

Mass of animal $\propto L^3$

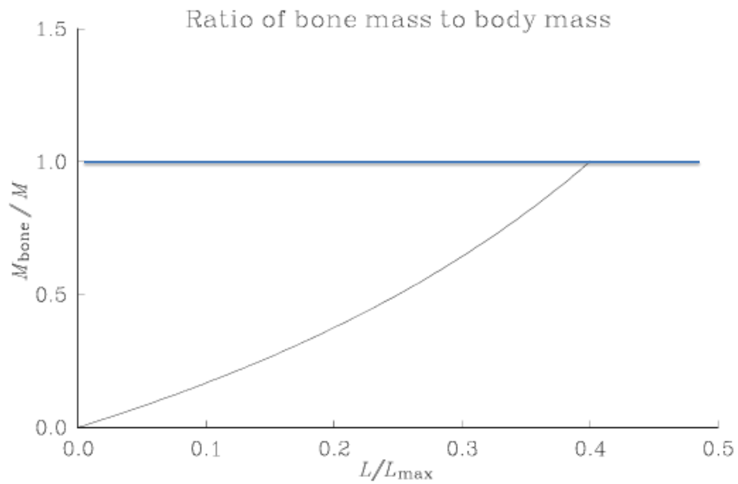
But strength of bones proportional to their cross-sectional area $\propto L^2$

The max force a muscle can exert in tension $\propto L^2$.



Strength to weight ratio (of proportionally scaled) animals scales as **L-1**

$$\frac{m_{bone}}{m_{tot}} = \frac{3}{2}\frac{L}{L_{max}}\frac{1}{1-L_{max}}$$



Therefore larger animals change their proportions
– **larger bone and muscle diameter**

Max 60 tons, dinosaur

[J. V Noble, Physics of the human body, Virginia tech course](https://wiki.epfl.ch/scaling/documents/Papers/scaling_animals.pdf)
https://wiki.epfl.ch/scaling/documents/Papers/scaling_animals.pdf

Scaling of tree trunk diameter vs. tree height

Scaling Relationships

Allometric relationship: Height vs. diameter in trees

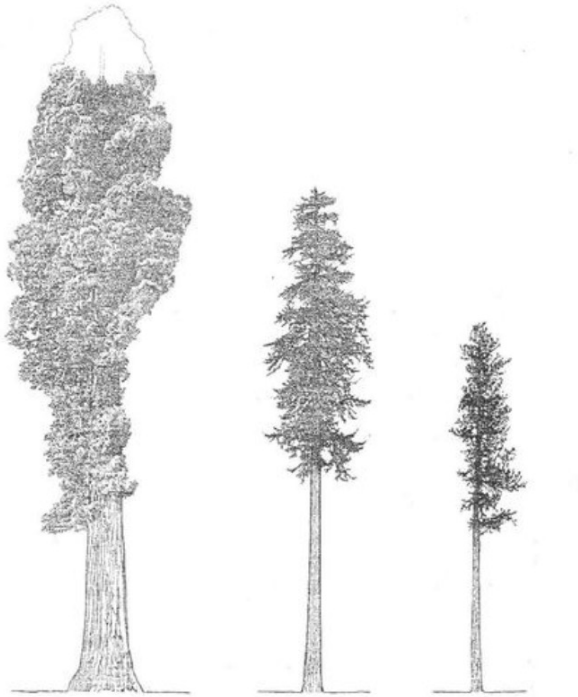
The critical buckling height for cylinders is:

$$H_{critical} = k * (E/\rho)^{1/3} * D^{2/3}$$

Therefore, if trees maintain "**elastic similarity**":

$$H \propto D^{2/3}$$

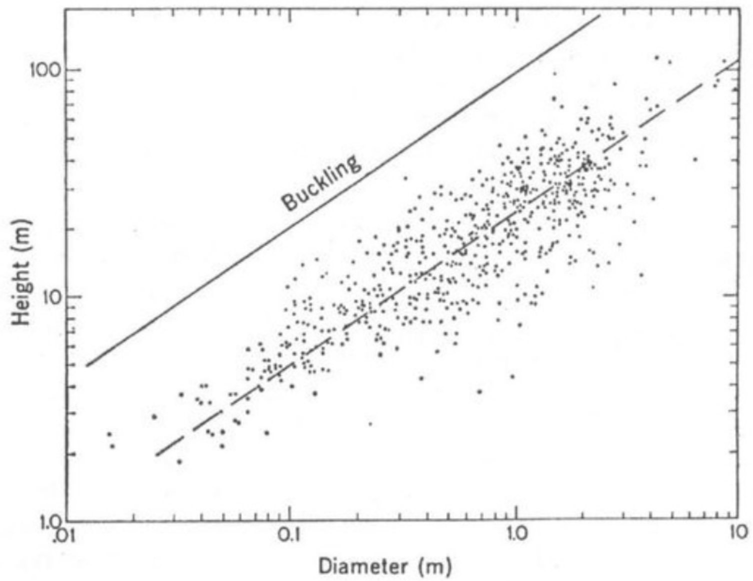
$$D \propto H^{3/2}$$



Giant sequoia | Douglas fir | Ponderosa pine

See Greenhill (1881); Figure from McMahon (1975)

Dataset for U.S. record trees.

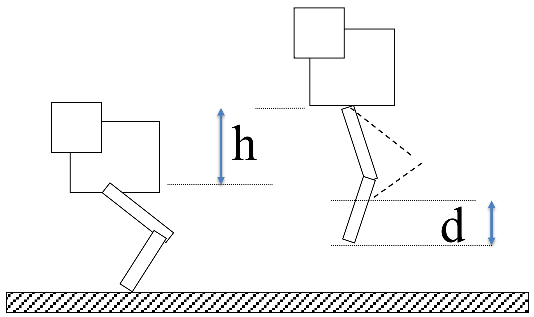


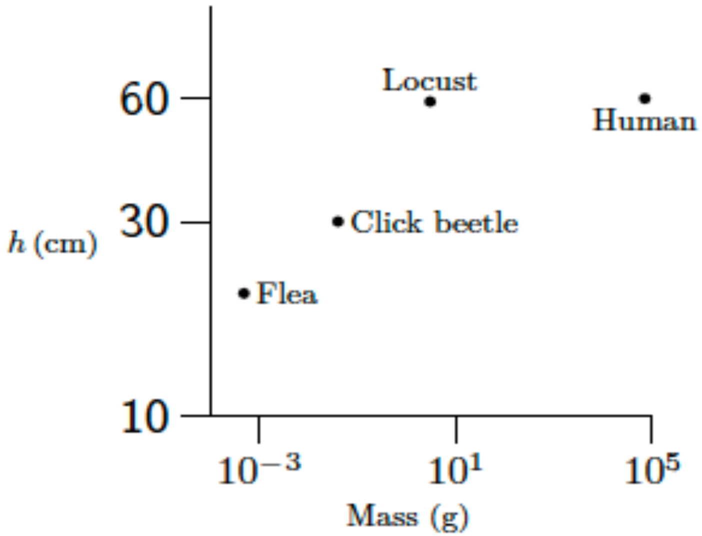
Both lines have slopes = 2/3; the broken line is 1/4 the magnitude of the complete line

Trees avoid buckling under their own weight, with a 4x safety factor

https://slideplayer.com/slide/12278649/

Scaling in nature: when size does not matter





from "Scaling: Why Animal Size is So Important", Knut Schmidt-Nielsen, Cambridge University Press; 1984

- Compute max Jumping height h
- Assume bone critical stress σ_{cr}
- d: elongation T: force from muscles

$$T_{\max} = \sigma_{\max} \cdot r^2 \qquad T_{\max} d = Mgh$$

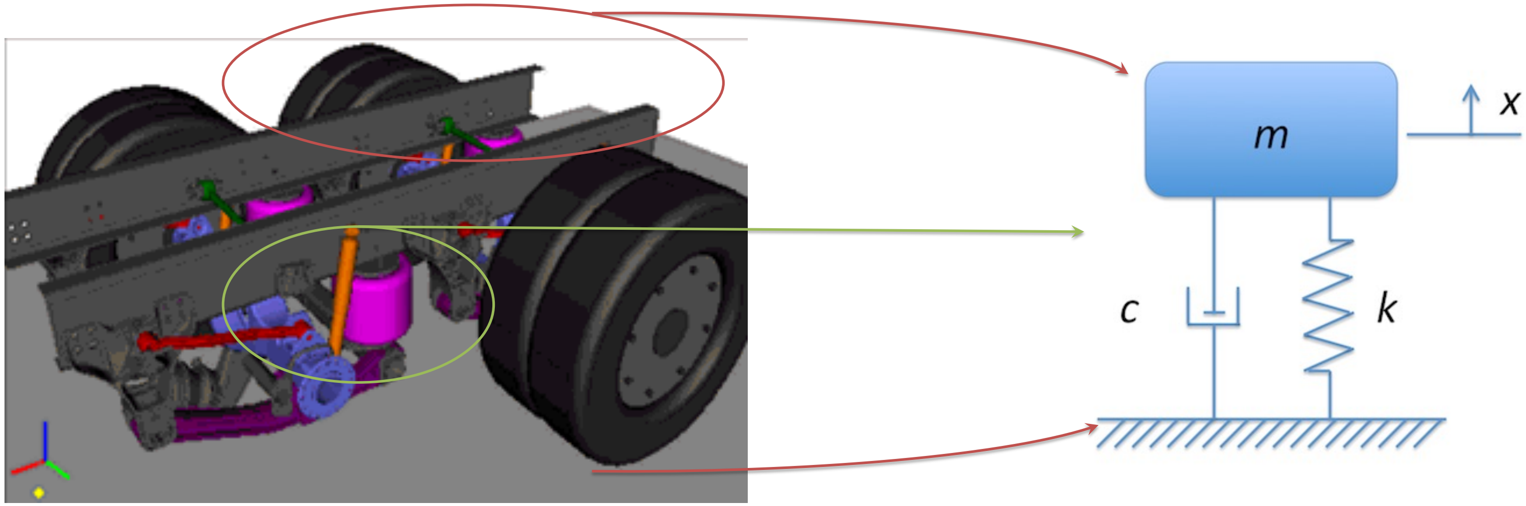
- If $r \propto L$ then $h = \frac{T_{\max} d}{Mg} \propto \frac{L^2 L}{L^3} \propto L^0$

Max jumping height is independent of size !!

- If $r \propto L^{1.5}$ then $h = \frac{T_{max} d}{Mg} \propto \frac{L^2 L^{1.5}}{L^3} \propto L^{0.5}$

2. Intro to Lumped Element Modelling

Mechanical Example



- What did we neglect?
- What did we gain ?
- Genius or stupid approach?

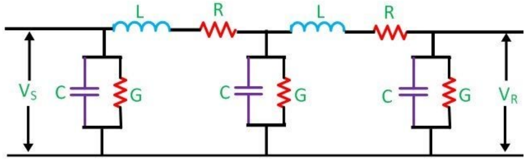
Lumped Element Model (LEM) allows simplifying systems

- A lumped element model (LEM) is a way to simplify a given system into a combination of discrete elements (“lumped”)
  - The elements can exchange energy/information with other elements
  - Signals/Variables propagate instantaneously inside the elements

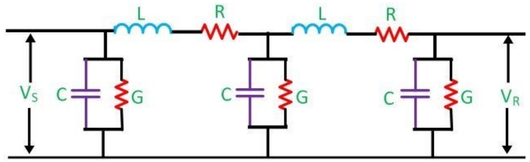
- This approach greatly reduces the complexity of the system

- In some cases, the model will give a good approximation (but can also give nonsense if physics poorly understood)

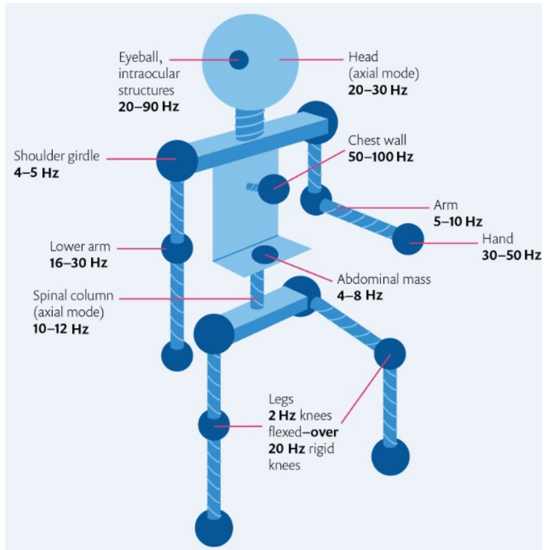
Lumped element = information travels instantaneously through device



Lumped element = information travels instantaneously through device



https://www.youtube.com/watch?v=7Ht5m2iwDys

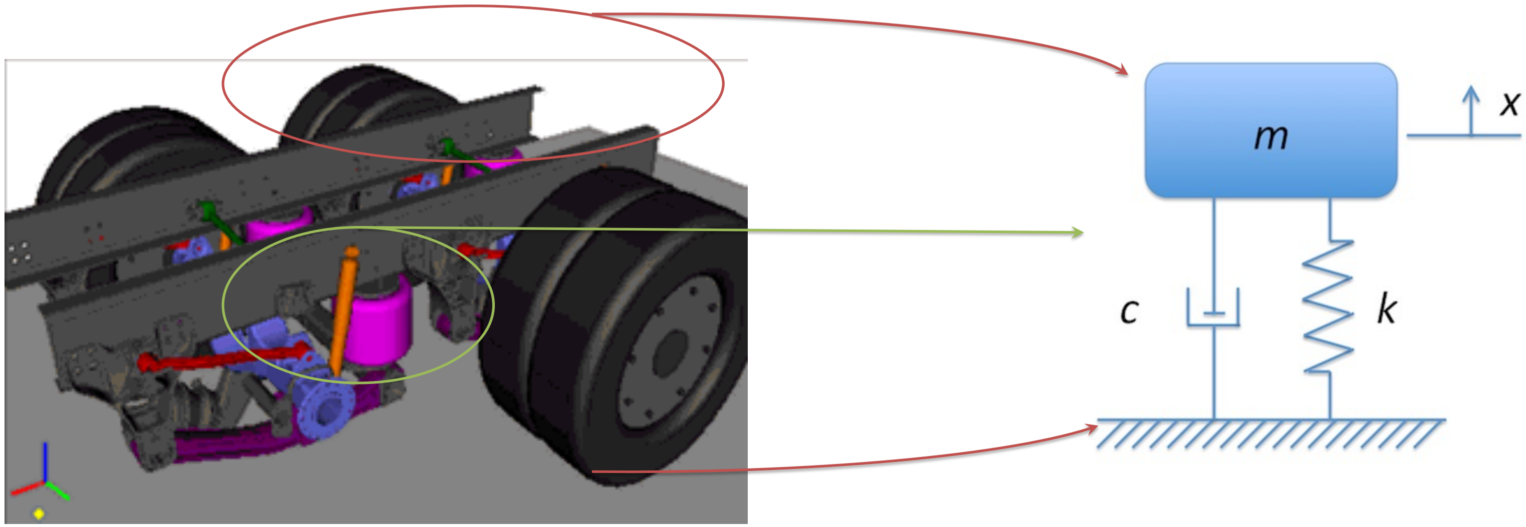


https://www.comsol.com/blogs/how-to-use-lumped-elements-to-model-a-mechanical-system/

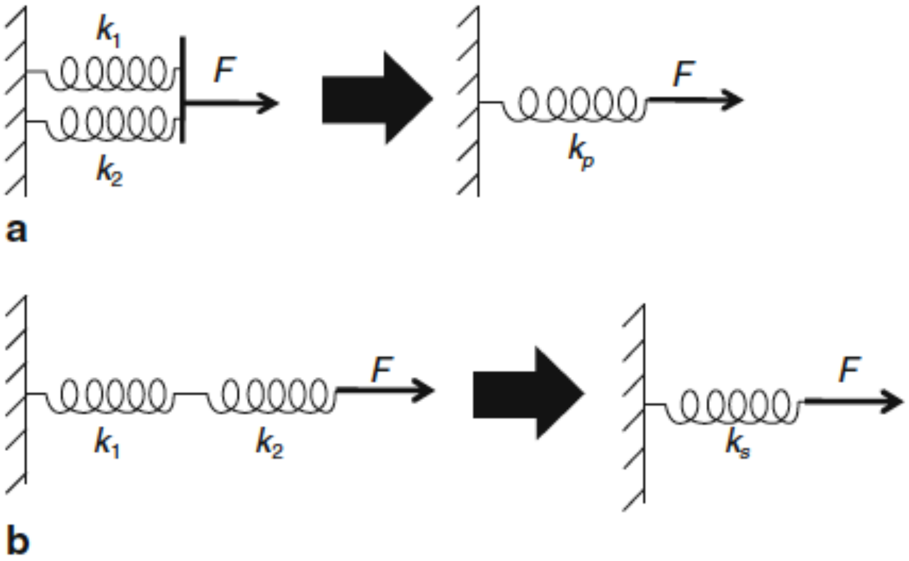
Mechanical Example

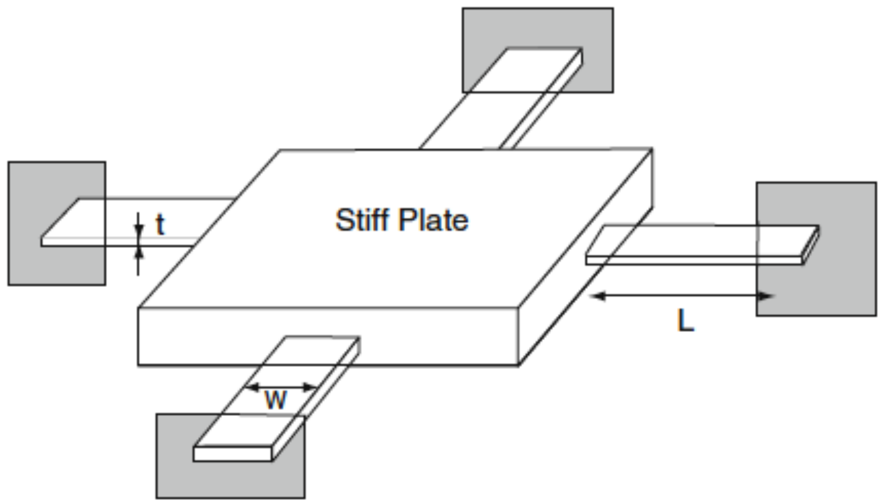
- Variables: position (x) and velocity ($\dot{x}$) of the mass
- Mass connected to ground via spring and damper
  - Approximations: Mass is rigid and concentrated; spring and damper are massless
- Eq of Motion: $m\ddot{x} + c\dot{x} + kx = F(t)$

Solve in Matlab, C++, or analytically
If make equivalent circuit, can use Circuit simulator



Complex springs can be modeled as combination of simple elements





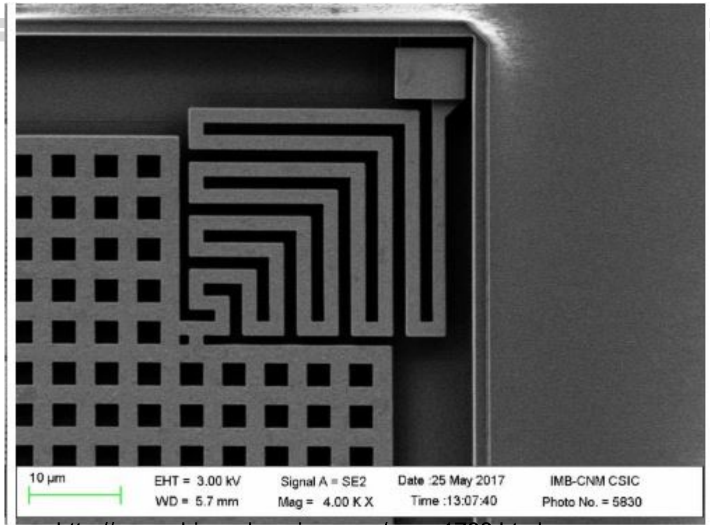
Springs are more complex than $F = kx$…

- Simple Cantilever has
  - Many bending modes, torsion, etc.
  - Anchor effects
  - Material non-linearities
  - Depends on how load is applied

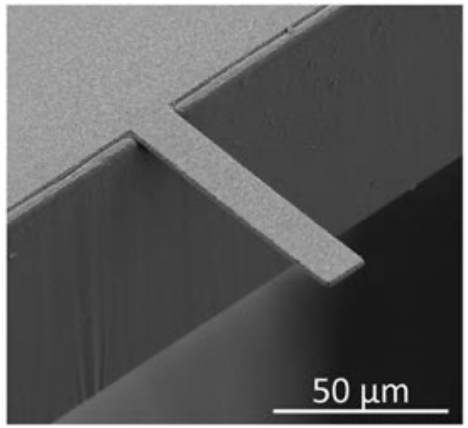
  - **Yet in Lumped Element Modeling (LEM), we will represent a complex deformation as a simple spring!**

  - **LEM is powerful** as we includes key cantilever dimensions as parameters, allowing quick design of full system, with analytical or numerical solutions

  - **But a useful LEM requires some physical insight!**



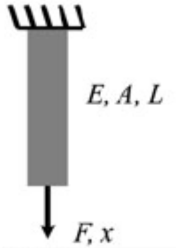
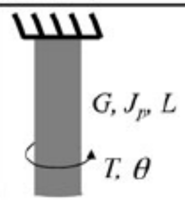
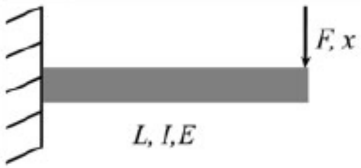
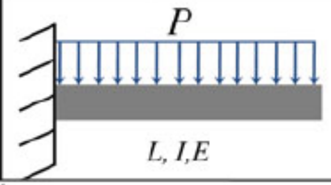
http://www.chipscalereview.com/news1708.html



http://microchem.com/Appl-MEMs-Cantilevers.htm

Several different *k* for a simple cantilever, depending on how it is deformed…

Table 4.1 The linear stiffness coefficient k of some of the common structure configurations in MEMS. The unit of k for all loading types is force/length. For wide beams ($b > 5h$) of cases 3–9, replace E with $E/(1-\nu^2)$, where ν is the Poisson's ratio. When calculating the natural frequency of a flexible structure alone, use an "effective" mass for the structure

1) Axially loaded bar		$k = \frac{EA}{L}$ E: Modulus of elasticity A: Area of cross section L: Length of bar
2) Rod under torque		$k = \frac{GJ_P}{L}$ J_p: Polar moment of inertia of the cross section G: Shear modulus L: Length of rod
3) Cantilever beam under point load at the tip		$k = \frac{3EI}{L^3}$ E: Modulus of elasticity I: Moment of inertia of the cross section L: Length of bar
4) Cantilever beam under uniformly distributed pressure		$k = \frac{8EI}{L^3}$

For one spring, many spring constants, depending on:

- how beam bends, and
- where force is applied.

Younis, M. I. *MEMS Linear and Nonlinear Statics and Dynamics*. **20,** (Springer US, 2011)

Lumped-parameter modeling in mechanics

- the dynamics of mechanical structures can be reduced to a mass-spring model where the natural pulsation is given by

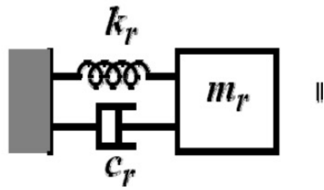
$$\omega_0 = \sqrt{k / m}$$

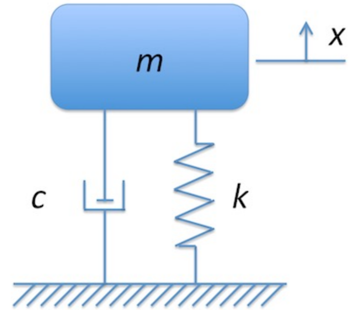
- k is given by calculation such as on previous slide

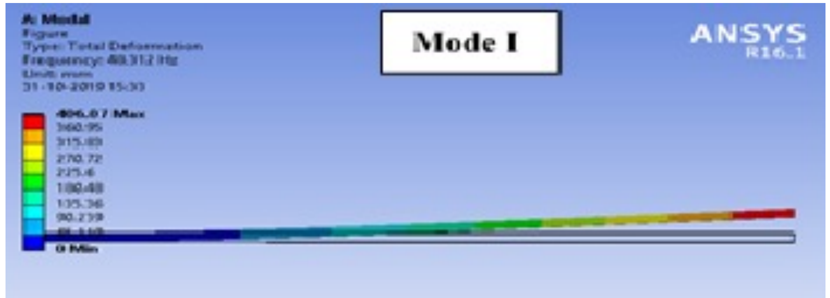
- But what mass to use? Seems obvious, but ….

- For each mode, we define an **effective mass,** m^*, that gives the appropriate natural frequency for a given structure of stiffness k.

- m^* is smaller than m for modes where entire mass is not moving







see: “Elements of Lumped-Parameter Modeling in MEMS”, in M. I. Younis, MEMS Linear and Nonlinear Statics and Dynamics

Effective mass for a cantilever (point load at tip)

- exact analytical calculation of the natural freq. of a cantilever: $$f_0 = \frac{1}{2\pi}\frac{h}{l^2}\sqrt{\frac{E}{\rho}}$$

$$\omega_0 = \sqrt{k/m}$$

E: young's modulus
ρ: density
h: thickness
l: length
w: width

- natural frequency in equivalent effective mass-spring model: $$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m*}}$$

- By taking the stiffness of the cantilever for end loading: $$k = \frac{Ewh^3}{4l^3} \quad \propto L$$

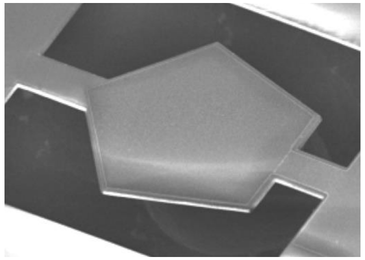
- Effective mass of 1st vibration mode of the cantilever : $$\boxed{m^* = \frac{1}{4}\rho \cdot w \cdot h \cdot l \quad = m/4}$$

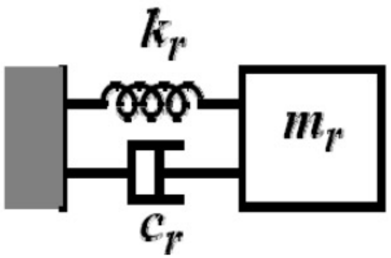
see: Elements of Lumped-Parameter Modeling in MEMS, in M. I. Younis, MEMS Linear and Nonlinear Statics and Dynamics

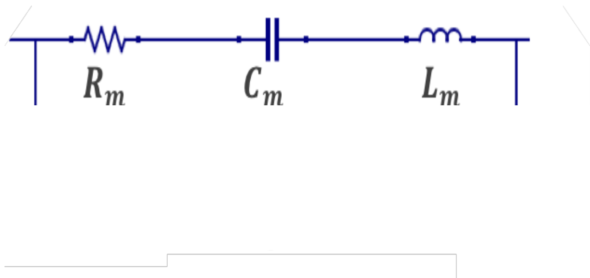
Electrical equivalent circuit

- Very often, we will work in the electrical domain, as very easy to solve electrical circuits (using p-spice, QUCS, etc)

Mechanical Variable	Electrical Variable
Damping, c	Resistance, R
Stiffness^{-1}, k^{-1}	Capacitance, C
Mass, m	Inductance, L
Force, f	Voltage, V
Velocity, v	Current, I





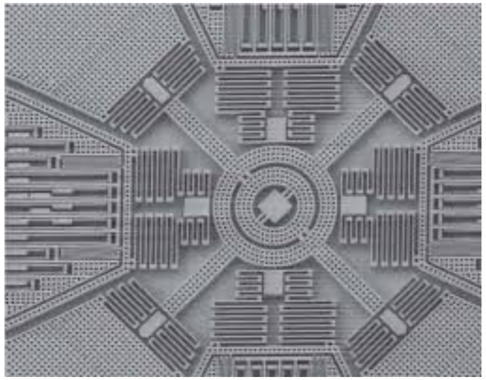


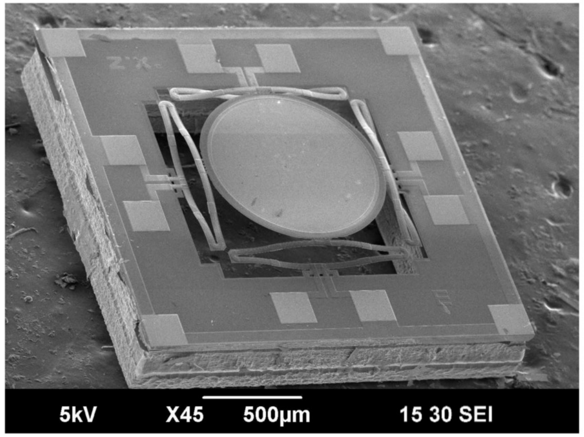
Equivalent circuit – Generalized variables

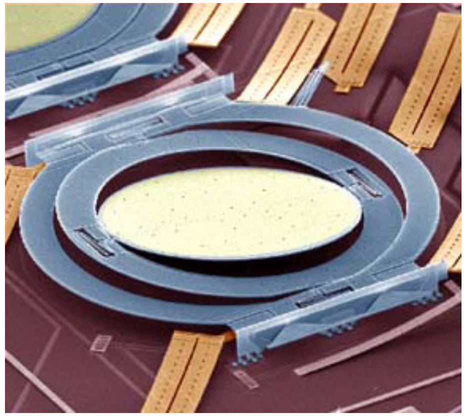
Generalized	Electrical case	Mechanical case	Fluidic case	Thermal case
Displacement – q	Charge – q	Displacement – x	Volume – V	Heat – Q
Flow – f	Current – I	Velocity – v	Flow rate – Q	Heat flow – $\dot{Q}$
Effort – e	Voltage – V	Force – F	Pressure – P	T difference – ΔT
Momentum – p	–	Momentum – p	Momentum – Γ	–
Resistance – R	Resistor – R	Damper – c	Fluid resist. – R	Thermal resist. – R
Capacitance – C	Capacitor – C	Spring – $1/k$	Fluid capac. – C	Thermal capac. – C
Inertance – L	Inductor – L	Mass – m	Inertance – m	–
Node law	Kirchoff's	Continuity of space	Mass conservation	Heat conservation
Mesh law	Kirchoff's	Newton's 2^{nd}	Relative P	Relative T

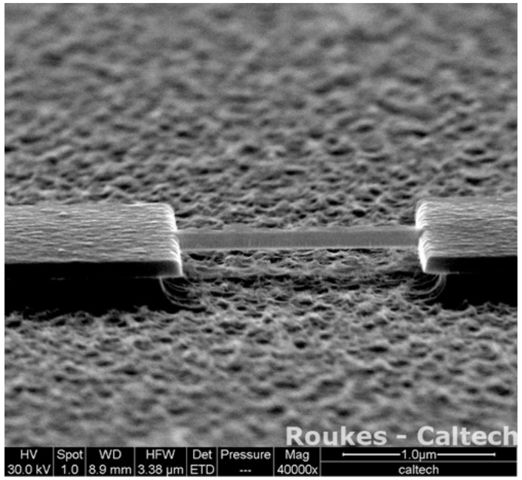
3. Cantilevers

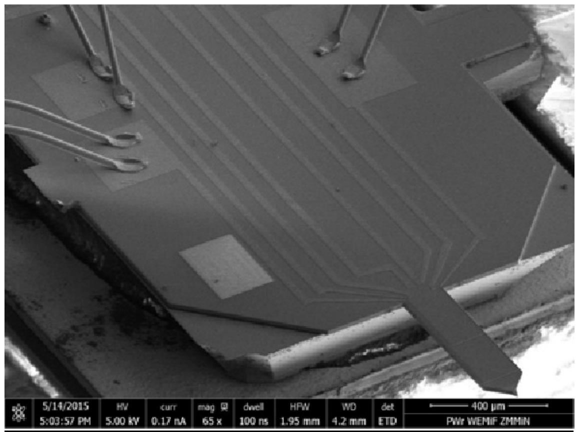
Cantilevers = Essential spring building block of MEMS

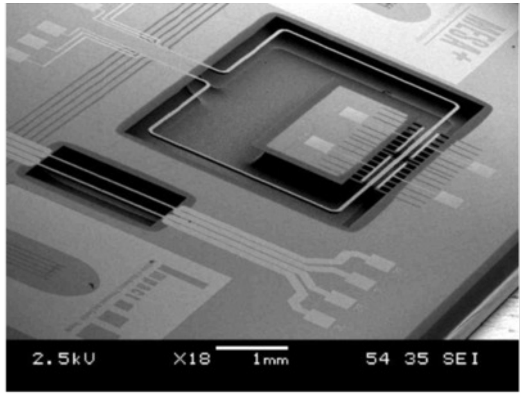












Cantilever: stress, strain and gravity

- Surface stress (at clamping edge) $$\sigma = \frac{6l}{wh^2} \cdot F$$

- Maximal allowed force F_{lim} at tip before fracture $$F_{\mathrm{lim}} = \sigma_{\mathrm{max}} \frac{wh^2}{6l} \propto L^2$$

  where σ_{max} is fracture strain

Length l, width w, thickness h

- Deformation z_{lim} under F_{lim} $$z_{lim} = \frac{F_{max}}{k} = \frac{2\,\sigma_{max}}{3E}\frac{l^2}{h} \sim L$$

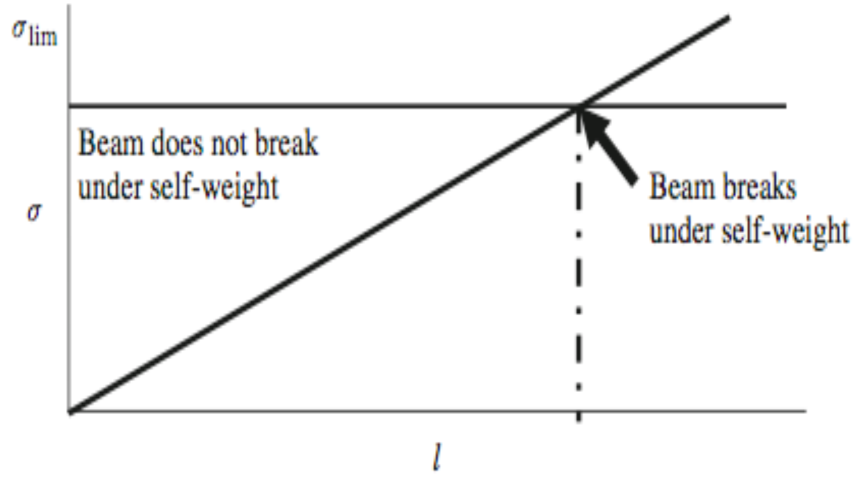
- Stress due to own weight (gravity) $$\sigma_x = \frac{\rho g l^2}{h} \propto L$$

Cantilever: stress, strain and gravity

- Beam bending due to gravity $$\Delta z_g = \frac{3}{2}\frac{\rho g}{E}\frac{l^4}{h^2} \propto L^2$$
- Gravity bending can be related to resonance frequency

$$\Delta z_g \cong \frac{0.38}{f_0^{\,2}}$$

- <u>Numerical example</u>:
  - Poly-Si cantilever 2x2 μm^2, l=1 mm
  - => Δz_g=50 nm σ_x=0.01 MPa
  - Stress for 1 μm deflection σ_x =72 MPa (compare to yield stress of polysilicon: 2 GPa)
  - Considering a yield stress of 2 GPa, the beam would survive 200'000 G shock! (not true, as other factors are overlooked)



R1: Mechanics Over Micro and Nano Scales, S Chakraborty, Springer 2011, p.72

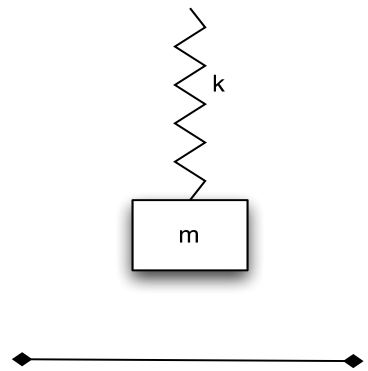
Accelerometer sensitivity (mass on a spring)

- To find the sensitivity, all we need to know is the resonance frequency !

$$S_x = \frac{x}{a} = \frac{m}{k} \propto L^2$$

- The mechanical sensitivity S_x to acceleration (x/a) can be fully determined by the resonance frequency, independently of mechanical parameters:

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad S_x = \frac{1}{\omega_0{}^2}$$



- Example: ADLX50 ω_0=24 kHz => according to simple formula S_x=1.7 nm/g (this also means that the sensing element of accelerometer must be very sensitive to displacement).
  - Measured mechanical sensitivity in ADXL50 d*x*/d*a* = 0.43 nm/g

- Bandwidth vs. Sensitivity. Smaller minimum acceleration requires bigger S_x (for fixed x_{min}), hence a slower device.

http://www.iris.edu/hq/instrumentation_meeting/files/pdfs/MEMS_Seismology.pdf

Cantilever dynamics (or why size matters)

EPFL

- First mode of cantilever: $f_1 = \frac{1}{2\pi}\omega_1 = \frac{1.03}{2\pi}\frac{h}{l^2}\sqrt{\frac{E}{\rho}} \qquad \propto L^{-1}$

- <u>Numerical example</u>: polysilicon cantilever (E=160 GPa) cantilever 2x2 μm^2
  - f_0=245 kHz for l=100 μm (first mode)
  - f_0=2.45 kHz for l= 1000 μm

Table 1: Fundamental Frequency *vs.* Geometry for *SiC*, [Si], and (GaAs) Mechanical Resonators

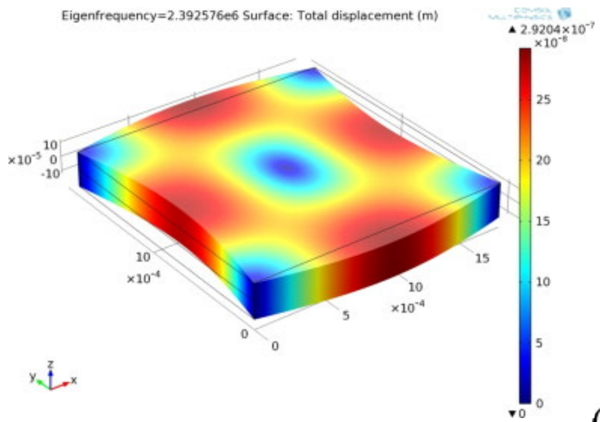
Boundary Conditions	**Resonator Dimensions** ($L \times w \times t$, *in* μm)			
	$100 \times 3 \times 0.1$	$10 \times 0.2 \times 0.1$	$1 \times 0.05 \times 0.05$	$0.1 \times 0.01 \times 0.01$
Both Ends Clamped or Free	*120 KHz* [77] (42)	*12 MHz* [7.7] (4.2)	*590 MHz* [380] (205)	*12 GHz* [7.7] (4.2)
Both Ends Pinned	*53 KHz* [34] (18)	*5.3 MHz* [3.4] (1.8)	*260 MHz* [170] (92)	*5.3 GHz* [3.4] (1.8)
Cantilever	*19 KHz* [12] (6.5)	*1.9 MHz* [1.2] (0.65)	*93 MHz* [60] (32)	*1.9 GHz* [1.2] (0.65)

"Nanoelectromechanical Systems", M. L. Roukes, Technical Digest of the 2000 Solid-State Sensor and Actuator Workshop, Hilton Head Isl., SC, 6/4-8/2000.

But: Mechanical energy vs. Thermal energy scaling …
Is there another way to get high frequencies?

Bulk vs. flexural modes

- We can also excite non-flexural modes: bulk modes!



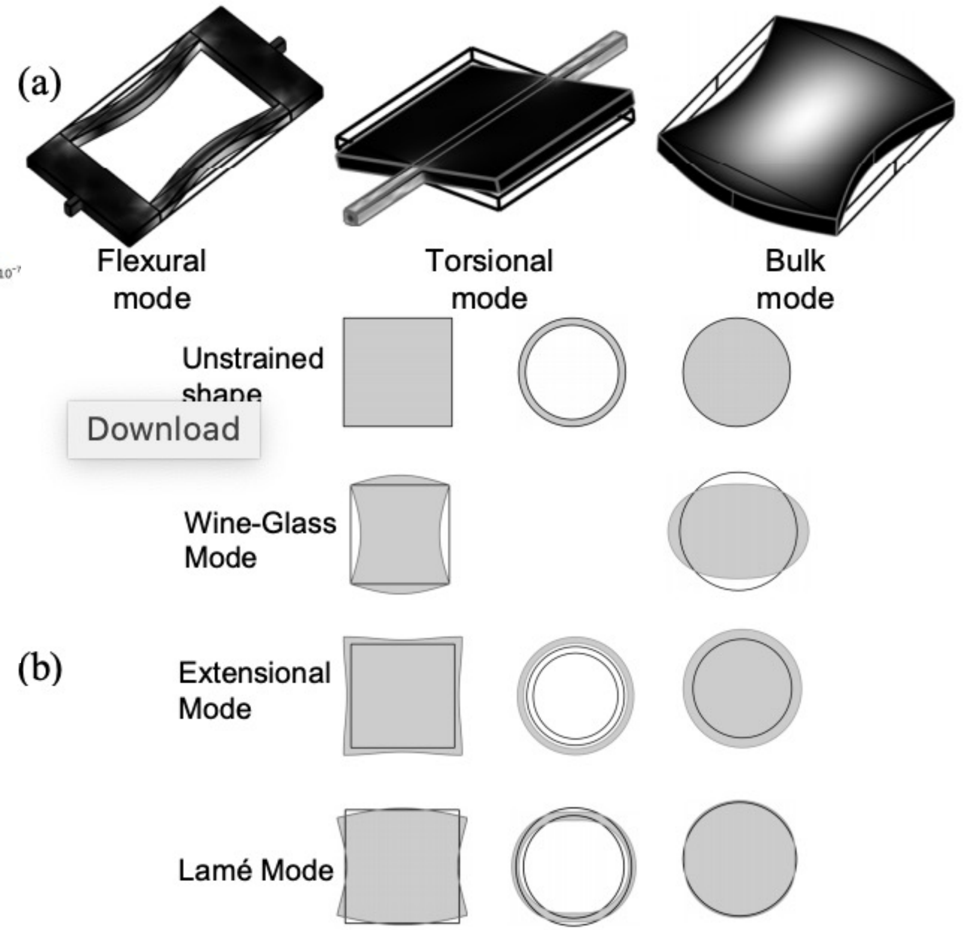


Figure 1: (a)Three basic types of resonators: Flexural, Torsional and Bulk mode structures (b) Commonly used Bulk mode resonator designs and various mode shapes

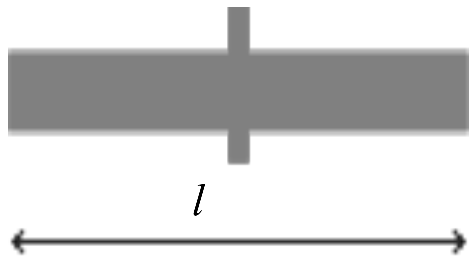
- Stored energy depends on moving mass: for bulk mode can make a thick and wide beams
- Bulk mode = Very stiff: get high frequency

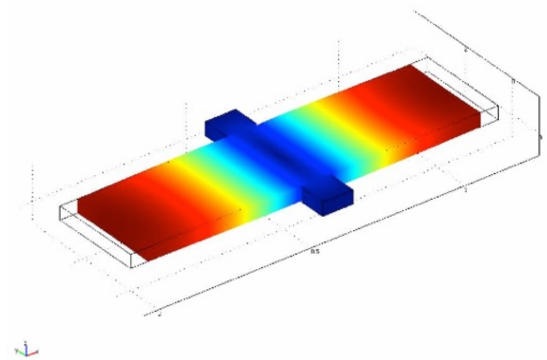
S. A. Chandorkar, M. Agarwal, R. Melamud, R. N. Candler, K. E. Goodson, and T. W. Kenny, "Limits of quality factor in bulk-mode micromechanical resonators," in *2008 IEEE 21st International Conference on Micro Electro Mechanical Systems*, doi: 10.1109/MEMSYS.2008.4443596.

With

Bulk modes: example of suspended bar

- Suspended bar, exciting longitudinal (not bending) vibrations, central suspension





- Fundamental mode: $$f_0 = \frac{1}{2l}\sqrt{\frac{E}{\rho}} \qquad \propto L^{-1}$$
  - Independent of thickness!
  - Independent of width!

- Example, Al beam with l= 60 µm, E=85 GPa, ρ=4500 kg/m^3
  f_1= 36 MHz f_3=108 MHz

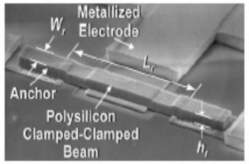
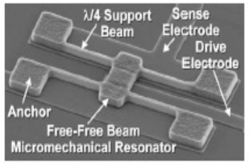
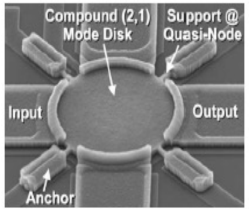
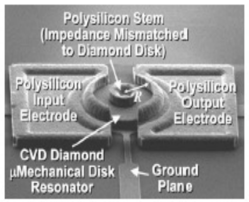
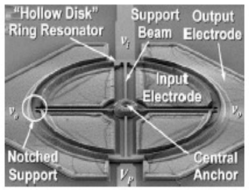
- Equivalent model $$f_0 = \frac{1}{2\pi}\sqrt{\frac{k^*}{m^*}} \qquad m^* = \rho \cdot w \cdot h \cdot \frac{l}{2} = m/2 \qquad k^* = \frac{\pi^2 Ewh}{2l}$$

$\propto L^{-1}$ $\propto L^3$ $\propto L^1$

Evolution of MEMS resonators to bulk modes

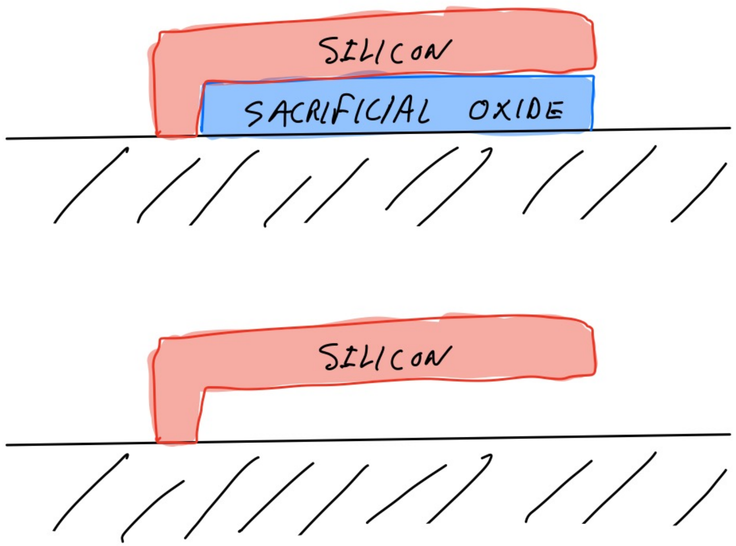
- Clark T.-C. Nguyen, “MEMS Technology for Timing and Frequency Control”, IEEE transactions on ultrasonics, ferroelectrics, and frequency control, Feb. 2007, p. 251

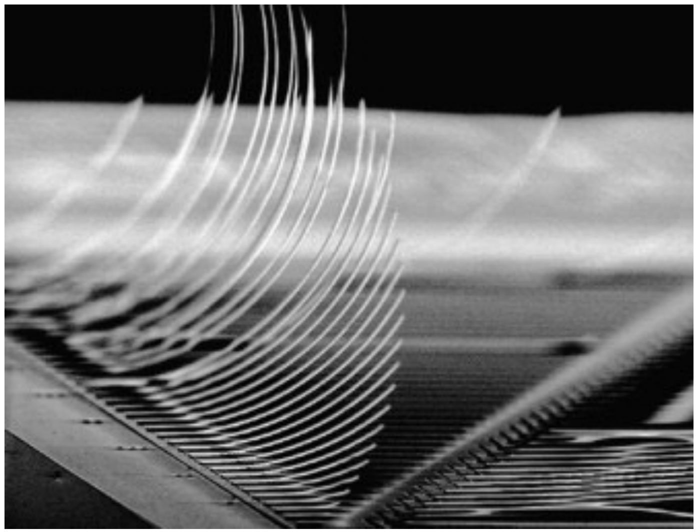
High Frequency Q Product Vibrating RF MEMS Devices.

Row	Resonator Type and Description	Photo	Performance
1	Clamped-Clamped Beam [19]: Flexural-mode beam fixed to the substrate at both ends. Micron-scale (i.e., 2-μm-thick) version is simple and works well below 30 MHz. Anchor losses reduce Q as frequency increases beyond 30 MHz. At right: 40-μm-long 7.8-MHz beam.		Demo'ed: $Q \sim 8,000$ @ 10 MHz (vac) $Q \sim 50$ @ 10 MHz (air) $Q \sim 300$ @ 70 MHz (anchor diss.) Q drop w/freq. limits freq. range Series Resistance, $R_x \sim 5$–$5,000\ \Omega^*$
2	Free-Free Beam [20]: Beam supported at flexural-mode nodal locations by quarter-wavelength torsional supports that “virtually levitate” the device, suppressing losses to anchors. Q remains high in vacuum as frequencies increase past 100 MHz. At right: 14.3-μm-long 82-MHz beam.		Demo: $Q \sim 28,000$ @ 10–200 MHz (vac) $Q \sim 2,000$ @ 90 MHz (air) No drop in Q with freq. Freq. Range: >1 GHz; unlimited w/scaling and use of higher modes Series Resistance, $R_x \sim 5$–$5,000\ \Omega^*$
3	Wine-Glass Disk [21]: Disk vibrating in the compound (2,1) mode. Can use either a center stem or perimeter supports. With quarter-wavelength perimeter supports located at radial nodal locations, achieves the highest Q's of any VHF on-chip resonator. At right: 26.5-μm-radius 73-MHz disk with perimeter supports.		Demo'ed: $Q \sim 161,000$ @ 62 MHz (vac) $Q \sim 8,000$ @ 98 MHz (air) Perimeter support design nulls anchor loss to allow extremely high Q Freq. Range: >1 GHz w/scaling Series Resistance, $R_x \sim 5$–$5,000\ \Omega^*$
4	Contour-Mode Disk [6], [22]: Disk vibrating in the radial-contour mode supported by a stem located at its center nodal point. Use of a material-mismatched stem maximizes the Q, allowing this design to set the record in frequency-Q product for any on-chip UHF resonator at room temperature. At right: 10-μm-diameter 1.5-GHz (in 2[nd] mode vibration) CVD diamond disk.		Demo'ed: $Q \sim 11,555$ @ 1.5 GHz (vac) $Q \sim 10,100$ @ 1.5 GHz (air) Balanced design and material mismatching anchor-disk design nulls anchor loss Freq. Range: >1 GHz; unlimited w/scaling and use of higher modes Series Resistance, $R_x \sim 50$–$50,000\ \Omega^*$
5	Spoke-Supported Ring [7]: Ring supported by spokes emanating from a stem anchor at the device center. Quarter-wavelength dimensioning of spokes nulls losses to the stem anchor, allowing this design to achieve the highest Q's past 1 GHz of any on-chip resonator. At right: 51.3-μm-inner and 60.9-μm-outer radii ring that attains 433 MHz in its 2[nd] contour mode.		Demo'ed: $Q \sim 15,248$ @ 1.46 GHz (vac) $Q \sim 10,165$ @ 1.464 GHz (air) $\lambda/4$ notched support nulls anchor loss Freq. Range: >1 GHz; unlimited w/scaling and use of higher modes Series Resistance, $R_x \sim 50$–$5,000\ \Omega^*$

Effect of Stress and Stress Gradients in cantilevers

- Due to fabrication processes, nearly all micromachined cantilevers have both **stress** and **stress gradients**



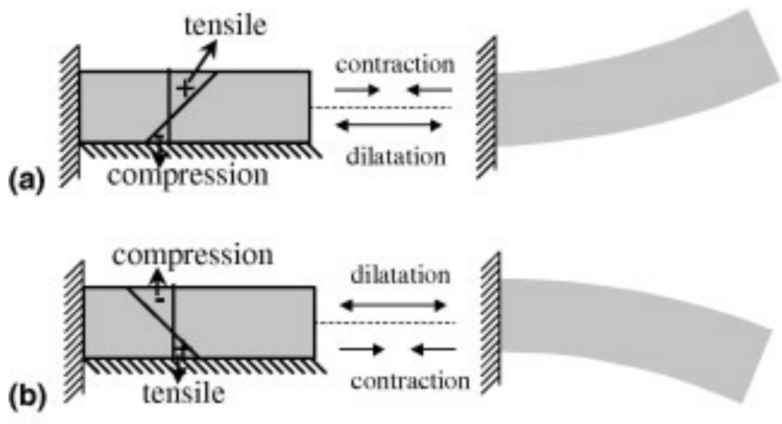


Witvrouw, A., Tilmans, H. A. C. & De Wolf, I. Materials issues in the processing, the operation and the reliability of MEMS. *Microelectronic Engineering* **76,** 245–257 (2004).

Stress and stress gradient in singly-clamped cantilever

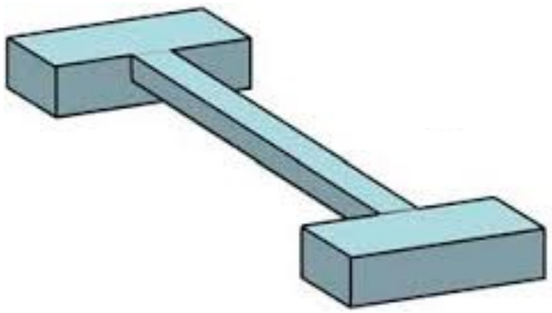
- Effect of Stress gradients depends strongly on cantilever clamping conditions

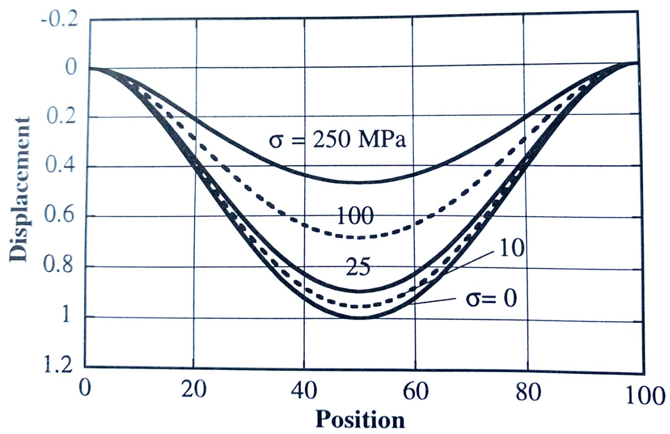
- Simple cantilever (single suspension)
  - Stress: only expansion or contraction (like Temperature change)
  - Stress gradient: large effect! Beam bending



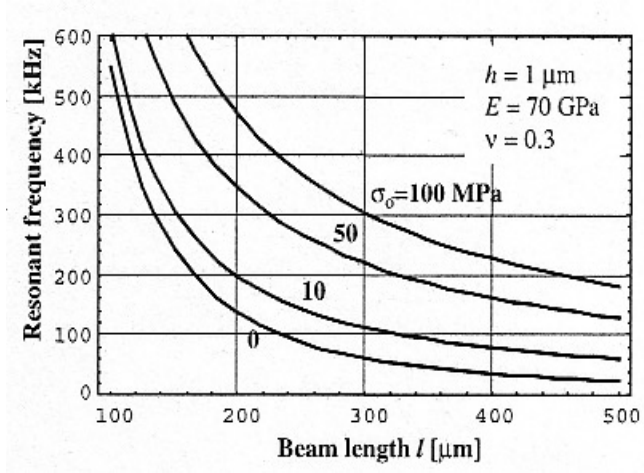
Stress and stress gradients: doubly-clamped cantilever

- Stress gradient
  - small effect
- Tensile stress
  - increase spring constant,
  - increase resonant frequency
- Compressive stress
  - buckling,
  - non-linear effective spring const.





Bending of a clamped silicon beam 100x2x2 µm³ under uniform load for different values of tensile stress.
S. Senturia, "Microsystem Design", p.232



Effect of tensile stress on the resonance frequency of doubly clamped cantilever

Standard test structures to measure stress in MEMS

- Guckel ring to measure tensile stress

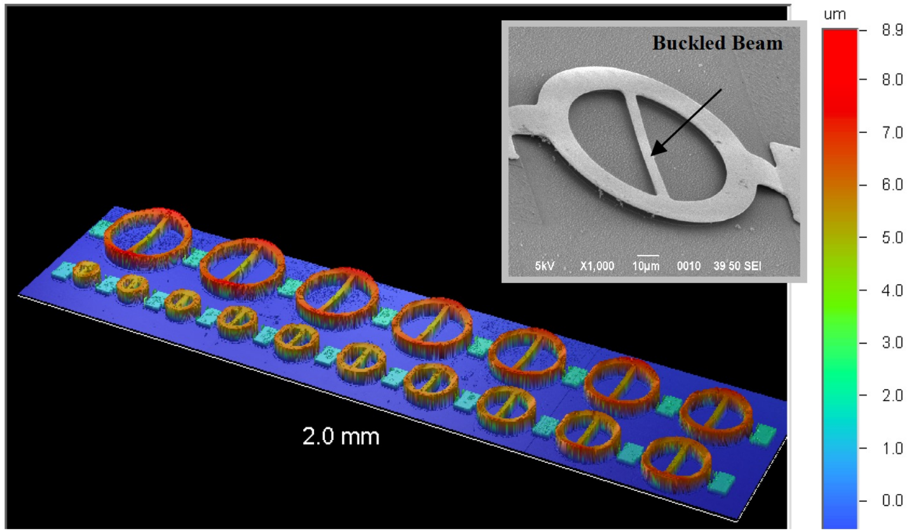
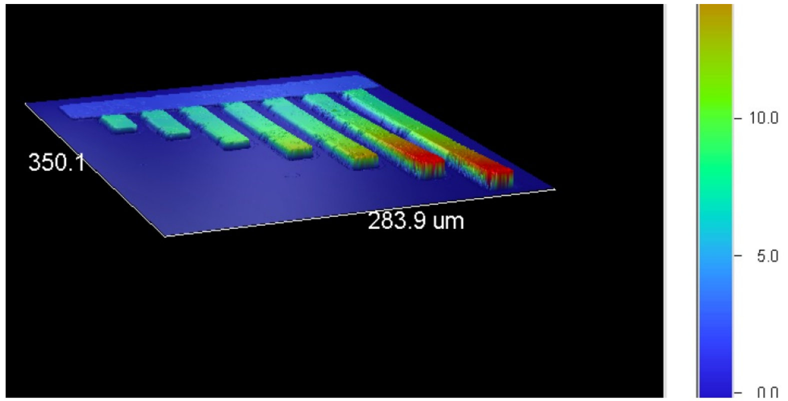
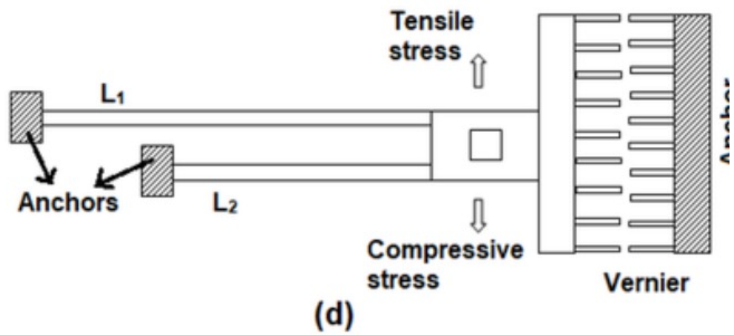


Fig. 7. 3D Optical profiler image of 2µm thick plated Au Guckel rings array with SEM micrograph of critical buckled central beam of R_c 60µm.

Sharma et al, Sensors & Transducers Journal, Vol. 13, 2011, pp. 21-30

- Cantilevers





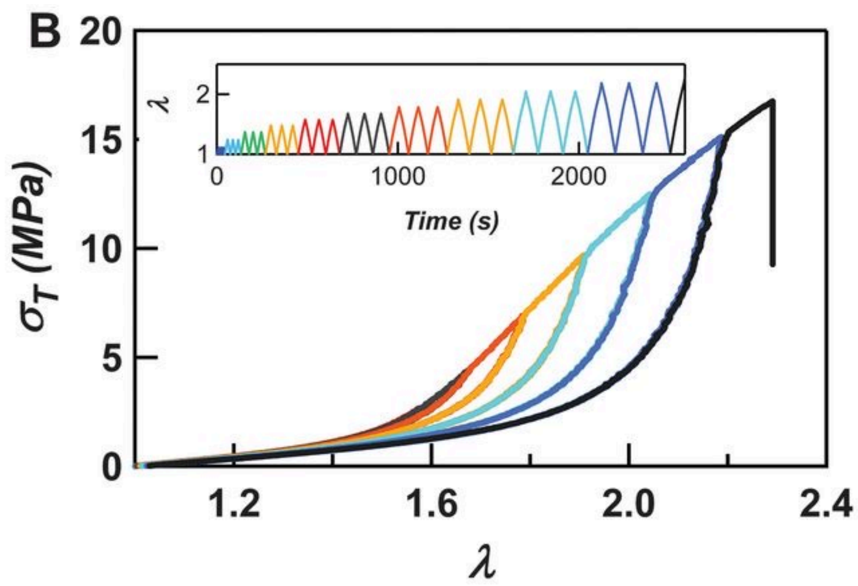
2021 paper: https://link.springer.com/article/10.1007/s10854-021-05405-8

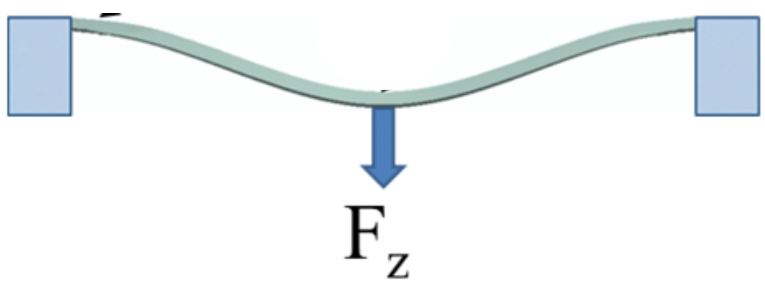
4. Non-linear effects in MEMS cantilevers

Non-linear effects in MEMS cantilevers

Non-linearities:

1. **Material** and 2. **Geometrical**

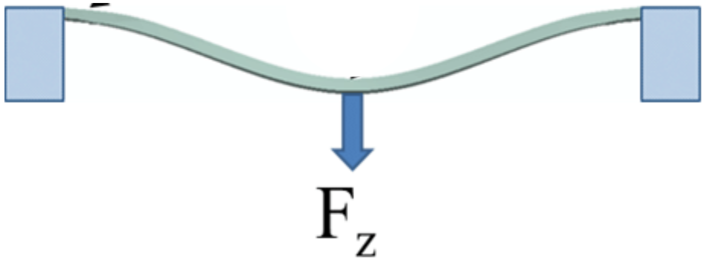




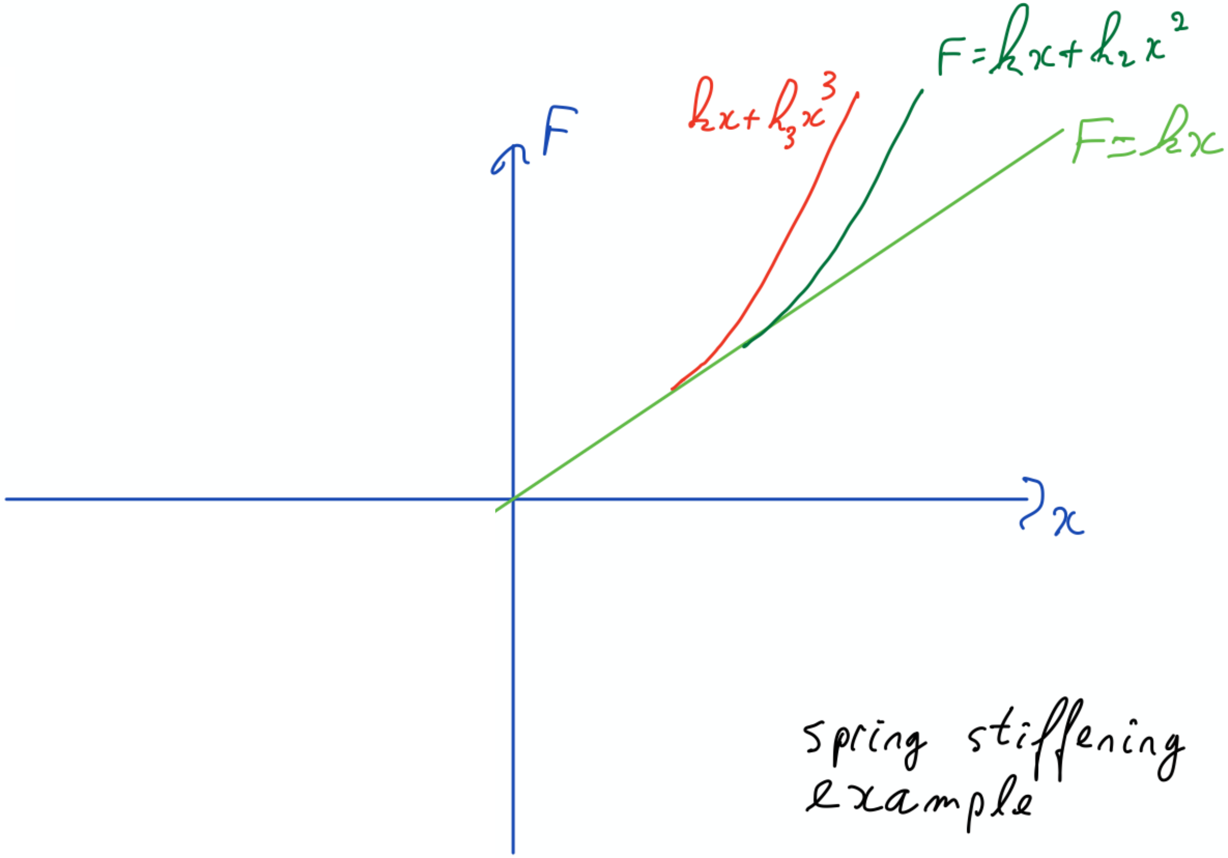
E Ducrot *et al*, Science 2014. DOI: 10.1126/science.1248494

Non-linear stress-strain
Hysteresis
Mullins effect…

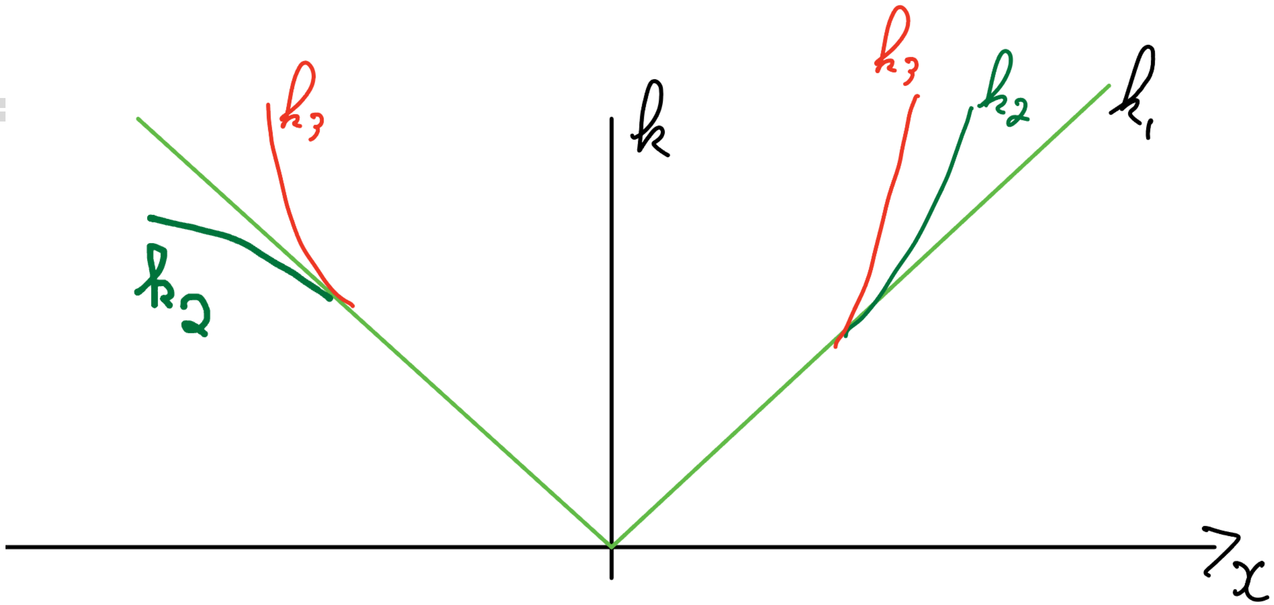
Spring constant for Geometrical non-linearities

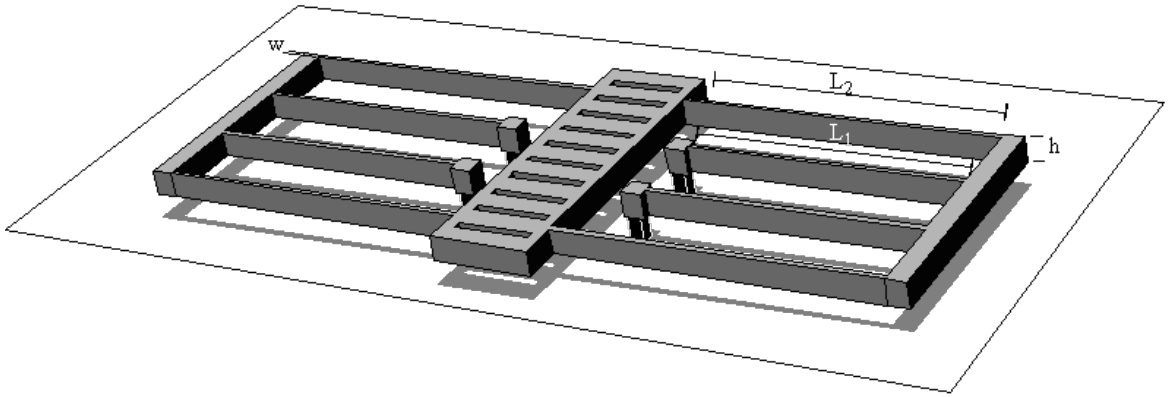


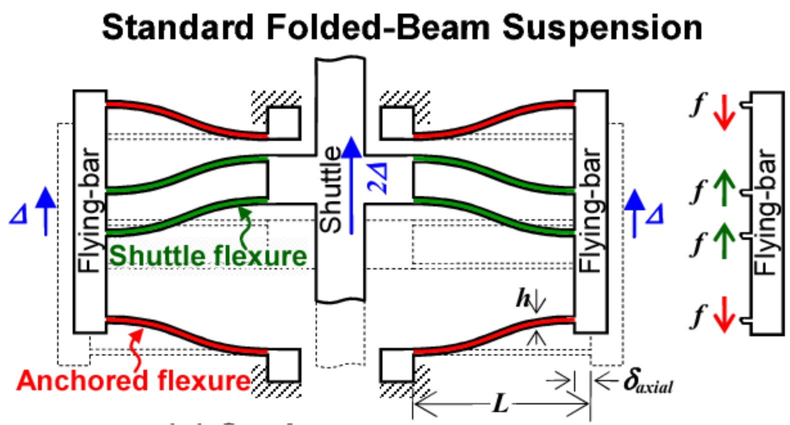
Bending → Bending + Stretching



How to express the non-linearity? Tailor expansion?







http://ieeexplore.ieee.org/stamp/stamp.jsp?tp=&arnumber=7050925

Non-linear effects in fixed guided beam: Very Important effect in MEMS!

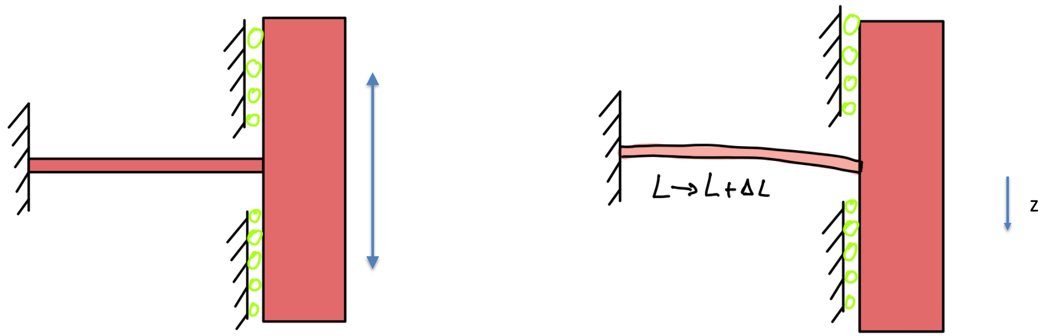
$$\boxed{F_z = k \cdot z + k_3 \cdot z^3}$$

$$k = \frac{Ew \cdot h^3}{l^3}$$

$$k_3 \cong \frac{252EA}{175l^3} = 1.4\frac{Ew \cdot h}{l^3} \propto \frac{1}{L}$$

$$\frac{k_3}{k} = 1.4\,\frac{1}{h^2} \propto L^{-2}$$

Non-linearity becomes more important as we scale down!



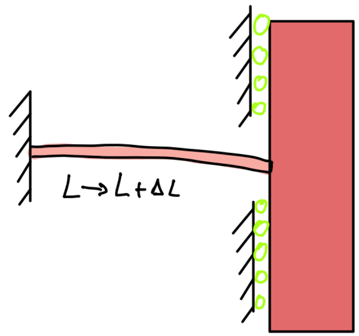
As mass is moved up or down, the spring bends… but it also stretches. So stiffness increases

Beam length *L*, thickness *h*, witdh *w*

$$\boxed{F_z = k \cdot z\left(1 + \frac{k_3}{k}z^2\right) = k \cdot z\left(1 + 1.4 \cdot \left(\frac{z}{h}\right)^2\right)}$$

Non-linear effects in fixed guided beam

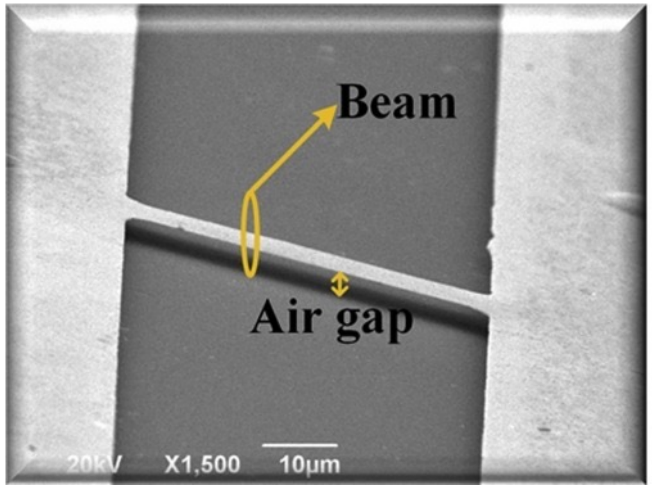
- important non-linearity when displacement *z* is a sizable fraction of thickness *h*
- a simple criterium: non-linear (10% effect) behavior when $z > 0.4h$
- Important consequences for MEMS resonators:
  - frequency stability: f_{res} depends on amplitude!
  - Limits power handling



$$k_3 \cdot z^3 > 0.1 \cdot k \cdot z$$

10%

When $z > 0.4h$



20 cm thick

40% motion = 8 cm

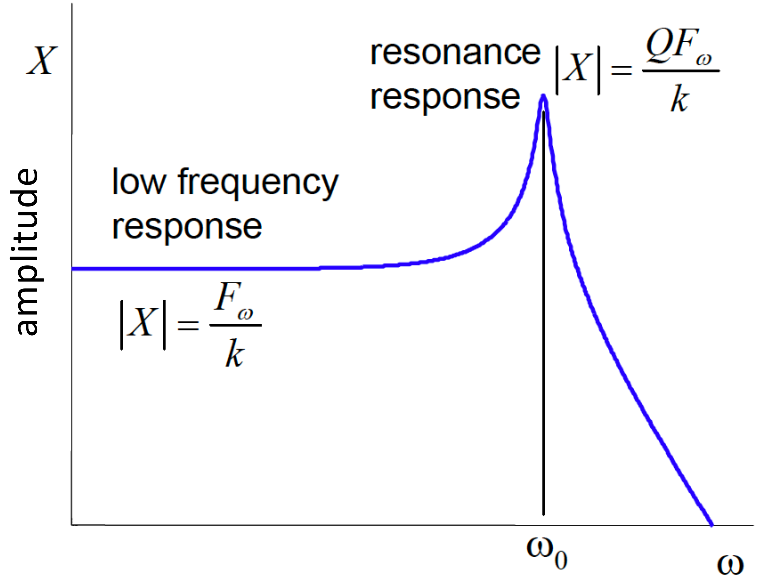
1 µm thick

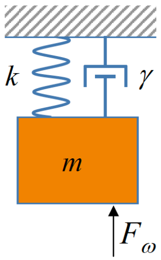
40% motion = 400 nm

Oscillator with non-linear spring: Duffing equation

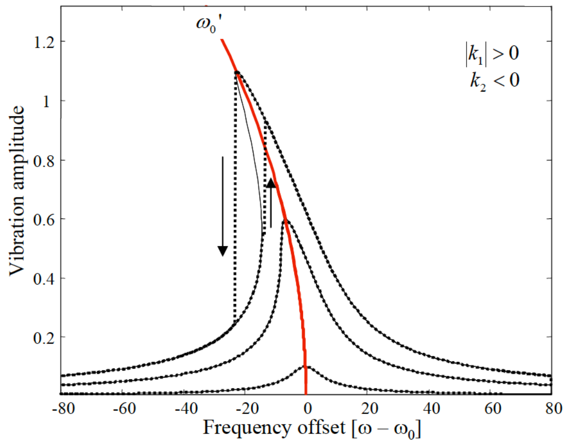
Good overview of Duffing eq. for MEMS:
http://www.kaajakari.net/~ville/research/tutorials/nonlinear_resonators_tutorial.pdf

$$\ddot{x} + 2\lambda\dot{x} + \omega_0{}^2 x + k_3 x^3 = F(t)$$

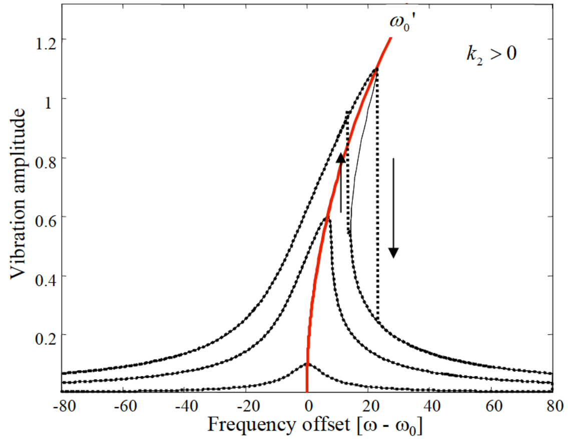




Linear case (ie k_3=0)



Spring softening (will return to this in electrostatic chapter)



Spring stiffening (mechanical)

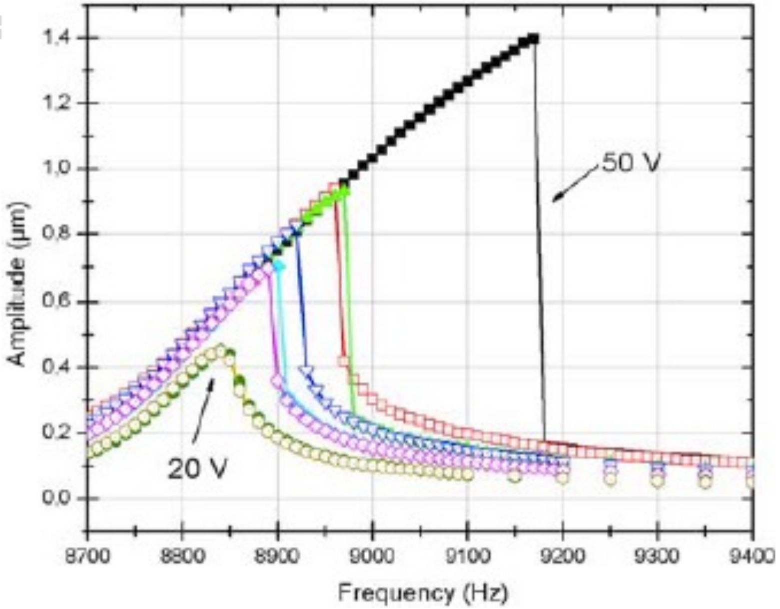
Duffing equation

- Duffing equation it has two solutions: One when sweeping frequency down and one when sweeping up !

- Approximate frequency shift for small amplitudes
  (z: displacement, h: thickness):

$$\frac{\omega^*}{\omega_0} = \sqrt{\frac{k^*}{k}} \approx \sqrt{1+1.4\cdot\left(z/h\right)^2} \qquad \frac{\Delta\omega}{\omega_0} \approx 0.7\cdot\left(z/h\right)^2$$

for 1% *h* displacement, 0.007 % relative frequency shift
for 10% *h* displacement (0.5 µm), 0.7 % relative frequency shift (do à do-dièse)



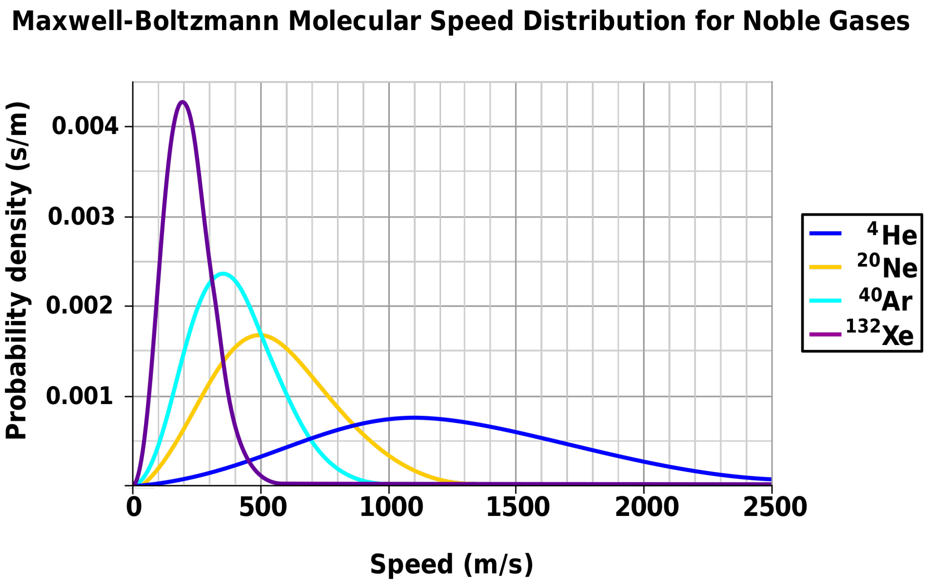
Test device: h=5µm, l=500µm, f_0= 8800
R. Guerre, EPFL-LMIS4

- Hysteresis rule: no hysteresis as long as: $|x| < \sqrt{\frac{8k}{3k_3Q}} \approx 0.7\cdot h\sqrt{\frac{1}{Q}}$
  for Q=10 x_{max}=0.22
  ie, only small displacements allowed for high Q devices if want to avoid hysteresis

5. Thermo-mechanical Noise

Thermo-Mechanical noise

Equipartition theorem: ½ k_BT for each DoF.



https://en.wikipedia.org/wiki/Equipartition_theorem

Probability density functions of the molecular speed for gases at 25° C

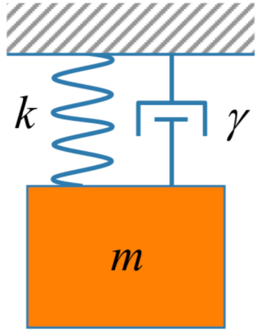
Ideal gas

$$v_{\mathrm{rms}} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3k_BT}{m}}$$

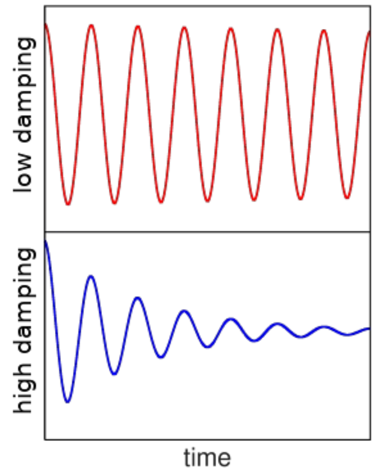
k_BT = noise power density in (W/Hz)

- How much does a MEMS cantilever weigh compared to a gas molecule?
- Is that a fair comparison?

Thermo-Mechanical noise



- Simple System = mass+ spring+ damper
- How is the mass thermally coupled to the world? Via damping element
- Damping = coupling



- No damping = mass decoupled from environment (no thermal exchange, no thermal noise, infinite Q)
- High damping = mass strongly coupled to the environment (lots of thermal exchange, high thermal noise, low Q)
- May seem a bit counter-intuitive: more damping, more thermal noise! Q$\sim L$
    - the issue is what frequency we consider: at resonance, or below resonance ???

Thermo-Mechanical noise

- Simple Harmonic oscillator exposed to a random fluctuating force F_n:

$$F_n(t,\lambda) = m\ddot{x} + \lambda\dot{x} + kx$$

λ=damping coefficient $\lambda = \frac{\omega_0 m}{Q} = \frac{\sqrt{k.m}}{Q} \sim L$

- Spectral density of fluctuating force F_n: (force-voltage analogy) $\boxed{F_{n,rms}(\omega) = \sqrt{4k_b T\lambda}}$ $\sim L^{0.5}$

$[N/\sqrt{Hz}]$

  - **The damping coefficient is crucial for allowing energy exchange between the oscillator and the medium** (both for coupling energy in and out of the oscillator)

  - Any mechanical system can be analyzed for mechanical-thermal noise by adding a force generator at each damping element.

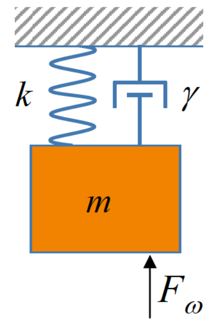
Good derivation in : http://www.kaajakari.net/~ville/research/tutorials/mech_noise_tutorial.pdf

Thermo-Mechanical noise, below the resonance frequency

- Position fluctuations (at frequencies well **below the resonance frequency**):

$$F = k \cdot x \longrightarrow x_{n,rms}(\omega) = \frac{F_{n,rms}(\omega)}{k} \qquad \sim L^{-0.5}$$

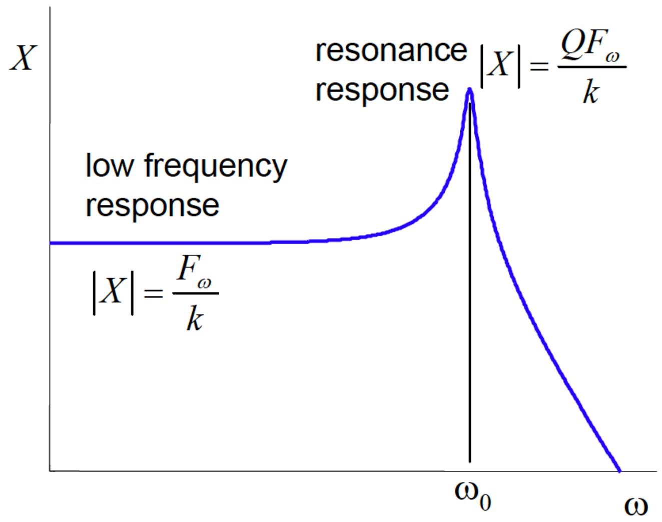
$$x_{n,rms}(\omega) = \frac{\sqrt{4k_b T\lambda}}{k} \quad \left[m/\sqrt{Hz}\right]$$ Spectral density



- **Mean** thermal displacement **for $\omega<\omega_0$** is:

$$x_{n,rms}\Big|_{f_1}^{f_2} = \frac{\sqrt{4k_b T\lambda}}{k} \cdot \underbrace{\sqrt{\Delta f}}_{\text{bandwidth}} \qquad [m]$$

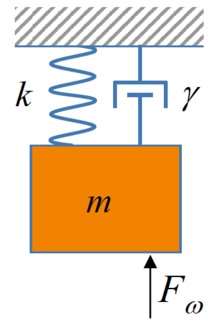
$$\sim L^{-1}$$



Thermo-Mechanical noise in accelerometer

$$S_x = \frac{1}{{\omega_0}^2}$$

EPFL



- Displacement due to external acceleration $x_a = \dfrac{ma}{k}$

- Minimal detectable acceleration a_{min} due to thermal noise: $a_{\min} = \sqrt{\dfrac{4k_b T\lambda}{m^2}}\sqrt{\Delta f}$
  ie when $x_a = x_{rms\_noise}$

- Scaling (assuming $\lambda \propto L$)

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \propto L^{-1} \qquad Q = \frac{\omega_0 m}{\lambda} \quad \propto L \qquad a_{\min} = \sqrt{\frac{4k_b T\omega_0}{mQ}}\sqrt{\Delta f} \qquad a_{\min} \propto L^{-5/2}$$

Typically a few tens of µg/sqrt (Hz) to mg/sqrt (Hz)

- The miniaturization of accelerometers dramatically reduces sensitivity (due to thermal noise)!
- The resolution can be improved by increasing Q, lowering ω_0 or increasing mass
- High Q element is however not desirable because it induce long oscillatory tail in signals …
- Same reasoning for pressure sensors, microphones, etc.

Key paper: T. Gabrielson, "Mechanical-thermal noise in micromachined acoustic and vibration sensors", IEEE Trans. on elec. Dev. Vol. 40, 5 (1993)

- Example (below resonance)

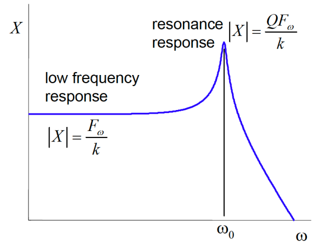
  - Silicon MEMS inertial mass 100 µm thick, 200 µm x 100 µm area

$$a_{\min} = \sqrt{\frac{4k_b T\omega_0}{mQ}}\sqrt{\Delta f}$$

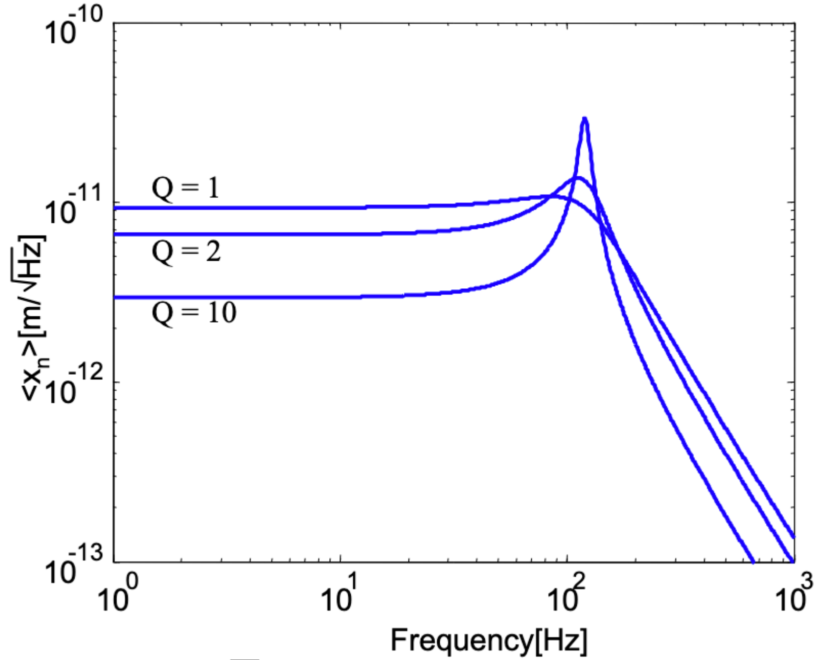
  - ω_0=20 kHz
  - Q =1
  - 1000 Hz bandwidth

- a_{min} = 10^{-2} $m.s^{-2}$
  = 1 milli-"g"

Thermo-Mechanical noise vs Q



https://www.kaajakari.net/~ville/research/tutorials/mech_noise_tutorial.pdf



$$< x_n > = \sqrt{\overline{x_n^2}}$$

$$x_{n,RMS}\Big|_{f_1}^{f_2} = \frac{\sqrt{4k_BT}}{k}\,\frac{\sqrt{\omega_0 m}}{\sqrt{Q}}\cdot\sqrt{\Delta f}$$

Below Resonance

Device dimensions: 1 mm x 1 mm x 0.2 mm
mass: $4.4 \cdot 10^{-7}$ kg
and spring constant: 0.25 N/m.

Quality Factor scaling

The quality factor is defined as:

$$Q = 2\pi \frac{energy\ stored}{energy\ lost\ per\ cycle}$$

Q for many loss factors scales as L

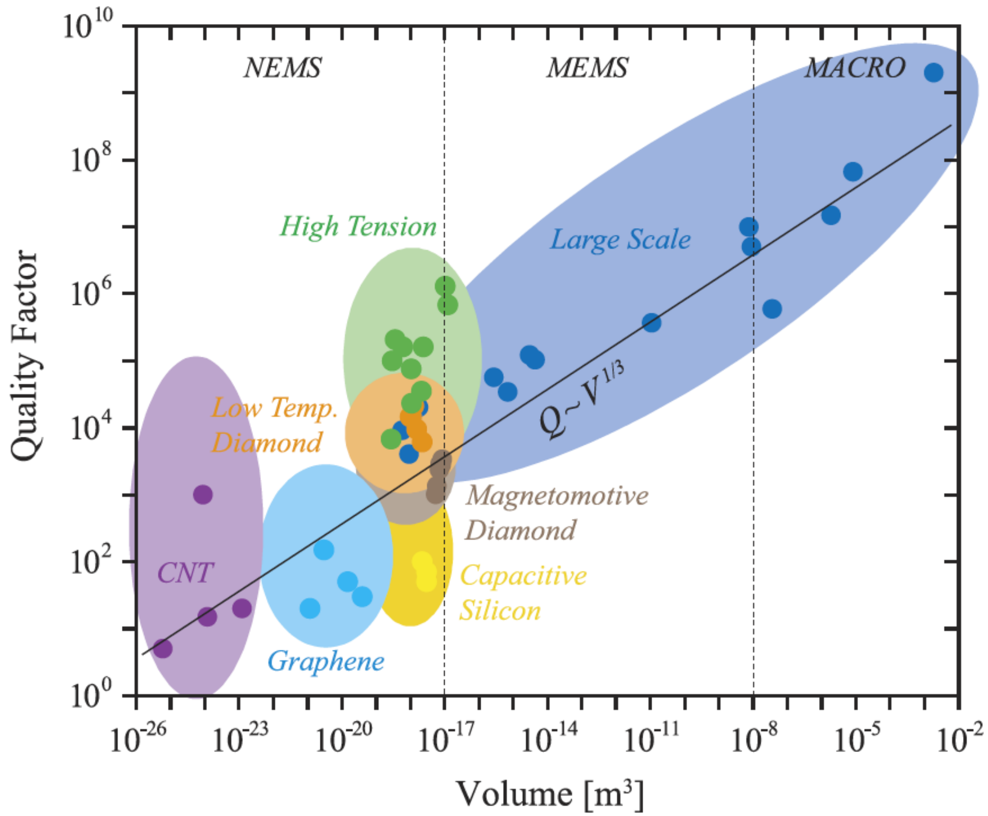
$$\frac{1}{Q} = \frac{1}{Q_1} + \frac{1}{Q_2} + \cdots = \sum_j \frac{1}{Q_j}.$$

Q factor depends on:

- Externally
  - Air damping
  - Clamping losses at supports
  - Coupling losses from transducers

- Internally
  - Thermo-elastic effects from defects in bulk, interfaces, fab-related damage, adsorbates on surface, …

- Q often depends on *surface to volume ratio* $\propto L$



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Physics Reports **534,** 89–146 (2014).